A Mixed Integral–Differential Operator and the Emergence of Algebraic–Oscillatory Structure

Abstract

We develop a symmetric operator combining derivatives and integrals. By analysing its action on exponential functions, we derive a palindromic characteristic equation that reduces to a quadratic invariant. The system splits into oscillatory and hyperbolic regimes, generating algebraic and transcendental constants.

1. Introduction

This work explores combining derivatives and integrals into a symmetric operator. Motivated by earlier sine-based systems, we investigate how structure emerges from balancing growth and inversion.

2. Operator Definition

$O[y] = a(D^2 + D^{\blacksquare 2})y + b(D + D^{\blacksquare 1})y + c\,y$. This operator is symmetric under $D \leftrightarrow D^{\blacksquare 1}$ and encodes forward/backward balance.

3. Exponential Response

For $y = e^{\wedge}(\lambda x)$, the operator reduces to $\Lambda(\lambda) = a(\lambda^2 + \lambda^{\blacksquare 2}) + b(\lambda + \lambda^{\blacksquare 1}) + c$. Thus exponentials are eigenfunctions.

4. Characteristic Equation

Solving $\Lambda(\lambda)=\mu$ yields a palindromic quartic: $a\lambda^{\blacksquare} + b\lambda^3 + (c-\mu)\lambda^2 + b\lambda + a = 0$.

5. Reduction

Using $t = \lambda + 1/\lambda$, we reduce the quartic to a quadratic: $a\,t^2 + b\,t + (c - \mu - 2a) = 0$.

6. Oscillatory Regime

If $|t| \leq 2$, then $\lambda = e^{\wedge}\{i\theta\}$ and $t = 2\cos\theta$. The system exhibits bounded oscillations.

7. Hyperbolic Regime

If $|t| > 2$, then $\lambda = e^u$ and $t = 2\cosh u$. The system exhibits exponential growth or decay.

8. Emergence of Constants

Examples include $\sqrt{13}$ and $\sqrt{5}$ derived from specific operator choices. These produce algebraic cosines and sines defining rotation angles.

9. Hyperbolic Constants

From earlier asymptotic work: $B_n \sim e^{(u/n)}$. These u values are logarithmic eigenvalues and satisfy hyperbolic balance equations.

10. Unified Structure

The invariant t unifies both regimes: $t = 2\cos\theta$ (oscillatory) or $t = 2\cosh u$ (hyperbolic).

11. Numerical Considerations

Direct quartic solvers introduce drift. Using the reduced equation preserves symmetry and exact structure.

12. Discussion

The operator links differential and integral calculus, generating structured spectra and constants.

13. Conclusion

The mixed operator provides a unified framework linking oscillation, growth, and algebraic structure.