A Mixed Integral–Differential Operator and the Emergence of Algebraic–Oscillatory Structure

Abstract
We study a symmetric operator combining derivatives and integrals. Its exponential spectrum reduces from a quartic to a quadratic invariant. The system naturally splits into oscillatory and hyperbolic regimes, producing algebraic and transcendental constants.

1. Operator Definition
$O[y] = a(D^2 + D^{-2})y + b(D + D^{-1})y + c\,y$

2. Exponential Response
For $y = e^{(\lambda x)}$, the operator reduces to $\Lambda(\lambda) = a(\lambda^2 + \lambda^{-2}) + b(\lambda + \lambda^{-1}) + c$.

3. Characteristic Equation
Solving $\Lambda(\lambda)=\mu$ yields a palindromic quartic: $a\lambda^4 + b\lambda^3 + (c-\mu)\lambda^2 + b\lambda + a = 0$.

4. Reduction
Let $t = \lambda + \lambda^{-1}$. Then the quartic reduces to: $a t^2 + b t + (c - \mu - 2a) = 0$.

5. Oscillatory Regime
If $|t| \leq 2$, then $\lambda = e^{i\theta}$ and $t = 2\cos\theta$.

6. Hyperbolic Regime
If $|t| > 2$, then $\lambda = e^u$ and $t = 2\cosh u$.

7. Hyperbolic Constants
From earlier work: $B_n \sim e^{u/n}$ with constants derived from hyperbolic balance equations. These correspond to logarithmic eigenvalues of the operator.

8. Unified Structure
$t = 2\cos\theta$ (oscillatory) or $t = 2\cosh u$ (hyperbolic). This unifies circular and hyperbolic geometry.

Conclusion
The operator provides a unified framework linking oscillation, growth, and algebraic structure.